3.9. Antiderivatives

DEF.: A function F is called an **antiderivative** of f on an interval I if

$$\boxed{F'(x) = f(x), \text{ for all } x \in I.}$$

If F is any antiderivative of f on I, then the most general antiderivative of f on I is

$$\boxed{F(x) + C}$$

where C is an arbitrary constant.

Denoted $\boxed{\int f(x)\,dx = F(x) + C}$

Why? Recall we proved by MVT that if $F_1'(x) = F_2'(x)$ for all $x \in I$, then F_1 & F_2 differ by a constant ($F_1(x) - F_2(x) = C$ for all $x \in I$).

Function	(Particular) Antiderivative
$cf(x)$	$cF(x)$
$f(x)+g(x)$	$F(x)+G(x)$
$x^n \ (n \neq -1)$	$\frac{x^{n+1}}{n+1}$
$\cos x$	$\sin x$
$\sin x$	$-\cos x$
$\sec^2 x$	$\tan x$
$\sec x \cdot \tan x$	$\sec x$

1. $\int (28x^3 - 21x^2 + 4x - 4)\,dx$

$$= 28 \cdot \frac{x^4}{4} - 21 \cdot \frac{x^3}{3} + 4 \cdot \frac{x^2}{2} - 4x + C$$

$$= \boxed{7x^4 - 7x^3 + 2x^2 - 4x + C}$$

2. $\int (4x^9 + 5\sec x \tan x)\,dx$

$$= 4 \cdot \frac{x^{10}}{10} + 5\sec x + C$$

$$= \boxed{\frac{2x^{10}}{5} + 5\sec x + C}$$

3. $\int \frac{7 - 7x^7}{x^4}\,dx = \int \left(\frac{7}{x^4} - 7x^3\right) dx = \int (7x^{-4} - 7x^3)\,dx$

$$= 7 \cdot \frac{x^{-3}}{-3} - 7\,\frac{x^4}{4} + C = \boxed{-\frac{7}{3}x^{-3} - \frac{7}{4}x^4 + C}$$

4. A curve $y = f(x)$ defined for $x > 0$ goes through the point $(1,0)$ and the slope of its tangent line at $(x, f(x))$ is $\frac{5}{x^3} - \frac{5}{x^6}$ for $x > 0$. $f(x) = ?$

DIFFERENTIAL EQUATION : Find $f(x)$ such that $\boxed{f'(x) = \frac{5}{x^3} - \frac{5}{x^6}} = \frac{dy}{dx}$

INITIAL CONDITION : $\boxed{f(1) = 0}$ $(y(1) = 0)$

$$\int\left(\frac{5}{x^3}-\frac{5}{x^6}\right)dx = \int\left(5x^{-3}-5x^{-6}\right)dx = 5\cdot\frac{x^{-2}}{-2}-5\cdot\frac{x^{-5}}{-5}+C$$

$$\Rightarrow f(x) = -\frac{5}{2}x^{-2}+x^{-5}+C$$

Determine C from the initial condition!

$x=1$

$$f(1) = -\frac{5}{2}\cdot 1^{-2}+1^{-5}+C = -\frac{3}{2}+C$$

but $f(1)=0$

$$\Rightarrow -\frac{3}{2}+C=0 \Rightarrow C=\frac{3}{2}$$

$$\boxed{f(x) = -\frac{5}{2}x^{-2}+x^{-5}+\frac{3}{2}}$$

5)
$$\int\left(\frac{3}{\sqrt[3]{x}}-3\sqrt[3]{x^2}\right)dx = \int\left(3x^{-1/3}-3x^{2/3}\right)dx$$

$$= 3\cdot\frac{x^{-1/3+1}}{-1/3+1}-3\,\frac{x^{2/3+1}}{2/3+1}+C = 3\cdot\frac{x^{2/3}}{2/3}-3\,\frac{x^{5/3}}{5/3}+C$$

$$= \boxed{\frac{9}{2}x^{2/3}-\frac{9}{5}x^{5/3}+C}$$

6) An object moves along a coordinate line w/ acceleration $a(t)=(t+3)^3$ m/s

a) The initial velocity is 5 m/s. $v(0)=5$

$a'(t)=v(t)$ $\quad v(t)=?$ $\quad \int a(t)\,dt = \int (t+3)^3 dt = \frac{(t+3)^4}{4}+C$

$$v(t) = \frac{(t+3)^4}{4}+C \qquad \text{but } v(0)=5$$

$$\Rightarrow v(0) = \frac{3^4}{4}+C = \frac{81}{4}+C \qquad \frac{81}{4}+C=5 \Rightarrow C=5-\frac{81}{4}=-\frac{61}{4}$$

$$\Rightarrow \boxed{v(t) = \frac{(t+3)^4}{4}-\frac{61}{4}}$$

b) Initial position is 4 m to the right of the origin. $s(0)=4$

$v'(t)=s(t)$ $\quad s(t)=?$ $\quad \int v(t)\,dt = \int\left(\frac{1}{4}(t+3)^4-\frac{61}{4}\right)dt = \frac{1}{4}\,\frac{(t+3)^5}{5}-\frac{61}{4}t+C$

Initial condition

$$\Rightarrow s(0) = \frac{1}{4}\,\frac{3^5}{5}+C = \frac{243}{20}+C = 4 \Rightarrow C=\frac{-163}{20}$$

$$\Rightarrow \boxed{s(t) = \frac{1}{20}(t+3)^5-\frac{61}{4}t-\frac{163}{20}}$$

(8) Initial Value Problem: $\frac{dy}{dx} = \frac{3}{X^5} + 6X^6$; $y(1) = -18$

$$y = \int (3X^{-5} + 6X^6)\,dX = 3\,\frac{X^{-4}}{-4} + 6\,\frac{X^7}{7} + C$$
$$= -\frac{3}{4}X^{-4} + \frac{6}{7}X^7 + C$$

$$\left.\begin{array}{l} y(1) = -\frac{3}{4} + \frac{6}{7} + C = \frac{3}{28} + C \\ \text{i.c.: } y(1) = -18 \end{array}\right\} \Rightarrow \frac{3}{28} + C = -18$$

$$C = -18 - \frac{3}{28}$$

$$\Rightarrow \boxed{y = -\frac{3}{4}X^{-4} + \frac{6}{7}X^7 - 18 - \frac{3}{28}}$$

(9) Solve: (IVT) $\frac{dr}{dt} = 3\cos(t)$; $r(-\pi/6) = 19$

$\Downarrow$

$$r(t) = 3\sin(t) + C$$

$\Downarrow$

$$\left.\begin{array}{l} r(-\pi/6) = 3 \cdot \frac{-1}{2} + C = -\frac{3}{2} + C \\ r(-\pi/6) = 19 \end{array}\right\} -\frac{3}{2} + C = 19$$

$$C = 19 + \frac{3}{2}$$

$$\boxed{r(t) = 3\sin(t) + 19 + \frac{3}{2}}$$